B.Sc. 5th Semester (Honours) Examination, 2019-20

MATHEMATICS

Course ID : 52112 **Course Code : SHMTH-502-C-12**

Course Title: Group Theory - II

Time: 2 Hours **Full Marks: 40**

The figures in the right hand side margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meanings.

1. Answer *any five* questions from the following: 2×5=10

(a) Define inner automorphism on a group.

(b) Show that every element of a commutator subgroup G' of a group G is a product of square.

(c) Is the group $\mathbb{Z} \times \mathbb{Z}$ cyclic? Justify your answer.

(d) Show that $Z(G)$, the centre of the group G, is a characteristic subgroup of G.

(e) Write down the class-equation of S_3 with justification.

(f) Let G be a group of order 15. Prove that G is cyclic.

(g) Give example of an infinite p-group where p is a prime.

(h) State Sylow's Third Theorem.

2. Answer *any four* questions from the following: 5×4=20

(a) Let G be a group. Then show that $G/Z(G)$ is isomorphic to $Inn(G)$ where $Inn(G)$ denotes the group of all inner automorphisms of G. 5

(b) (i) Define characteristic subgroup of a group.

(ii) Prove that every characteristic subgroup of G is normal in G.

(iii) Give example of a group G and a subgroup H of G such that H is normal in G but H is not a characteristic subgroup of G. 1+2+2=5

(c) Describe all abclian group of order 360. 5

(d) (i) Let G be a finite group and H be a proper subgroup of G of index n such that $|G| \nmid n!$. Then show that G contains a non-trivial normal subgroup.

(ii) Let G be a group of order 65 and H be a subgroup of order 13. Prove that G is not simple. 3+2=5

(e) Let G be a group of order p^2 where p is a prime. First show that $Z(G)$ non-trivial. Then prove that G is abelian. 2+3=5

(f) Use Sylow's Theorems to show that any group of order 36 is not simple. 5

3. Answer *any one* question from the following: 10×1=10

(a) (i) Prove that $Aut(\mathbb{Z}_8)$ is isomorphic to the Klein's Four Group. 2

(ii) Prove that the group $A \times B$ is abelian if and only if both of the groups A & B are abelian. 2

(iii) Let G be a group acting on a non-empty set S. Prove that the index of G_a, the stabilizer subgroup of a $(a \in s)$, in G is equal to the cardinality of the orbit of a. 4

(iv) Prove that any 5-Sylow subgroup of a group of order 45 is always normal. 2

(b) (i) Find all automorphisms of the group $(\mathbb{Z}_6 , +)$. 2

(ii) Is there any element of order 12 in the additive group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. 3

(iii) Let G be a group of order $105 = 3{\cdot}5{\cdot}7$. First show that any Sylow 7-subgroup and any Sylow 5-subgroup are normal in G. Then show that G has a cyclic subgroup of order 35. Finally show that if H is Sylow 7-subgroup, K is a Sylow 5-subgroup and L is a Sylow 3-subgroup of G then $G = HKL$. 1+1+1+2=5

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1. Answer *any five* questions: 2×5=10

(a) Prove that $Aut\ \mathbb{Z}_{10} \cong \mathbb{Z}_4$.

(b) Give an example of a subgroup which is not characteristic. Justify your answer.

(c) Show that the direct product $\mathbb{Z} \times \mathbb{Z}$ is not a cyclic group.

(d) How many elements of order 7 are there in a simple group of order 168?

(e) Prove that every group of order 15 is cyclic.

(f) Let G be a finite group that has only two conjugatie classes. Show that $|G| = 2$.

(g) Show that every group of order 45 has a normal subgroup of order 9.

(h) Prove that $Inn\ S_3 \cong S_3$.

2. Answer *any four* questions: 5×4=20

(a) (i) Show that for any group G, ${}^{G}/_{Z(G)} \cong Inn\ G$. 3

(ii) Find $Inn\ (D_4)$. 2

(b) Let G be a finite group. Let H be a subgroup of G of index p, where p is the smallest prime dividing $|G|$. Show that H is a normal subgroup of G.

(c) Define internal direct product of finite number of subgroups and show that if $G$is the internal direct product of its subgroups $H_1,\ H_2 \dots\dots\dots ,\ H_n$; then $G \cong H_1 \times H_2 \times \dots\dots\dots ,\ H_n$. 1+4=5

(d) Let G be a finite p-group with $|G| > 1$. Prove that $|Z(G)| > 1$, where $Z(G)$ is the centre of G.

(e) (i) Define Sylo p-subgroup.

(ii) Prove that no group of order 56 is simple. 1+4=5

(f) (i) Define group action with example. 1+1=2

(ii) Let G be a finite group and S be a finite G-set. Then prove that the number of orbits of S is $\frac{1}{|G|}\sum_{g \in G} F(g)$, where $F(g)$ is the number of elements of S fixed by g. 3

3. Answer *any one* question: 10×1=10

(a) (i) Let G be a group of order $2m$, where m is an odd integer. Show that G has a normal subgroup of order m. 4

(ii) Let G be a group and S be a G-set, then prove that for all $a \in s$, $[G: G_a] = |[a]|$. 3

(iii) Let H be a subgroup of a group G. Prove that if H has a finite index n, then there is a normal subgroup K of G with $K \subseteq H$ and $[G: K] \leq n!$. 3

(b) (i) Prove that $|G| = |Z(G)| + \sum_{a \notin Z(G)} [G: C(a)]$, where the summation runs over the complete set of distinct conjugacy class representative which does not belong to $Z(G)$. 3

(ii) State and prove Sylow's Second Theorem. 1+3=4

(iii) Let G be a finite group of order p^n where p is a prime and $n \geq 1$. Prove that any subgroup of G of order $p^{n-1}$is a normal subgroup of G. 3