B.SC. FOURTH SEMESTER (HONOURS) EXAMINATIONS, 2021

Subject: Mathematics **Course ID: 42111**

Course Code: SH/MTH/401/C-8 **Course Title: Riemann Integration and Series of Functions**

Full Marks: 40 **Time: 2 Hours**

The figures in the margin indicate full marks

Notations and symbols have their usual meaning

1. **Answer *any five* of the following questions:** **(2×5 = 10)**

a) Evaluate: $\lim_{x\to 3}\frac{1}{x-3}\int_3^x exp\left(\sqrt{1+t^2}\right)dt$, if it exists.

b) Examine the convergence of the improper integral $\int_{e^2}^{\infty}\frac{dx}{x\log(\log x)}$.

c) Let $f_n(x) = xe^{-nx}, x \geq 0$. Examine the uniform convergence of $\{f_n\}_n$.

d) Prove or disprove: "Given that f:[0,1] →[0,1] is Riemann integrable on [0,1], the function g on [0,1] defined by $g(x) = \frac{1}{2-f(x)}$, $x \in [0,1]$ is bounded and Riemann-integrable on [0,1]."

e) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n\, x^n$, where a_{2n}=$\frac{1}{3^n}$ and $a_{2n+1} = \frac{1}{3^{n+1}}, (n = 1,2,3, \ldots)$.

f) If $f(x) = \sum_{n=1}^{\infty}\frac{\sin nx}{n^2}$, then find $\lim_{x\to 0} f(x)$.

g) Discuss the applicability of the second mean value theorem to the integral $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \cos x\, dx$.

h) If f is bounded and integrable on [-π,π] and if a_n ,b_n are its Fourier coefficients , then show that $\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ converges.

2. **Answer *any four* of the following questions:** **(5×4 = 20)**

a) (i) Let $f: [0,1] \to \mathbb{R}$ be such that

$$f(x) = \begin{cases} \frac{1}{5^n}, & \text{for } \frac{1}{5^n} < x \leq \frac{1}{5^n}, \qquad n = 0,1,2,\ldots \\ 0 & for\ x = 0 \end{cases}$$

Evaluate $\int_0^1 f$, if the Riemann integral exists.

(ii) Evaluate: $\lim_{x\to 0}\frac{1}{x^2}\int_{x^2}^{x^4}\sin\left(\sqrt{t}\right)dt$. 3+2

b) (i) Obtain the Fourier series of f in $[-\pi, \pi]$, where $f(x) = x\, sinx$ and deduce that

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \frac{1}{7.9} + \cdots$$

(ii) Let $f_n(x) = xe^{-nx}, x \geq 0$. Examine the uniform convergence of $\{f_n\}_n$. 3+2

c) (i) Let $f_n(x) = \frac{nx}{1+nx}$, $x \in [0,1], n = 1,2,3,\ldots$ Is the sequence $\{f_n\}_n$ converges uniformly on [0,1]. Justify your answer.

(ii) Let $f_n(x) = \begin{cases} nx^2, & when\ 0 \le x \le \frac{1}{n} \\ x, & when\ \frac{1}{n} \le x \le 1\,; \end{cases}$ for n = 2,3,4,... Find the limit function of $\{f_n\}_n$ on [0,1]. 3+2

d) Prove that $\sum \frac{\sin n\theta}{n^p}$ converges uniformly for all values of $p > 0$ in an interval [α,2π-α), 0<α<π.

e) (i) Find $\lim_{x\to 0} \frac{\int_0^{x^2} sin\sqrt{t}\, dt}{x^3}$.

(ii) If $\sum a_n$ is an absolutely convergent series of real numbers, prove that $\sum \frac{a_n x^n}{1+x^{2n}}$ is absolutely convergent. 3+2

f) Prove that a power series can be integrated term by term on any closed and bounded interval contained within the interval of convergence.

3. Answer *any one* of the following questions: **(10×1= 10)**

a) (i) If $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ are both continuous on [*a*,*b*] and $\int_a^b |f-g|dx = 0$, then prove that $f(x) = g(x)$ on [*a*,*b*].

Also give an example of functions *f* and *g* both integrable on [*a*,*b*] such that $\int_a^b |f-g|dx = 0$, but *f*≠*g*.

(ii) Let $f\colon [0,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ x^2, & \text{if } x \text{ is irrational.} \end{cases}$

Is *f* Darboux integrable on [0,1]?

(iii) Show that $\log(1-x) + \log(1+x) + \log(1+x^2) + \log(1+x^4) + \cdots$ converges for $|x| < 1$. 4 + 3 + 3

b) (i) Show that $\frac{\pi^3}{24\sqrt{2}} < \int_0^{\frac{\pi}{2}} \frac{x^2}{sinx+cosx} dx < \frac{\pi^3}{24}$.

(ii) Examine the convergence of $\int_0^{\frac{\pi}{2}} sin^{m-1}x cos^{n-1}x\, dx$. 5+5

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